Goal: - compute the Br group for both sm. proj. Markov surface

Primary reference: Integral Hasse Principal and Strong Approximation for Markoff Surfaces
- Daniel Loughran, Vladimir Mitankin

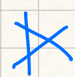
① Affine and Projective Markoff surface.

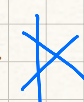
Eg: $U_m: x^2+y^2+z^2-xyz = m \subseteq \mathbb{A}^3$ $\qquad \pi_1(U_m(\mathbb{C})) = 0$?

meaning: if $\mathbb{A}^3$ is thought of as the character variety of F_2-rep in $SL_2(\mathbb{C})$ with coordinates $(\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB)$

then U_m is the set where $\mathrm{tr}(ABA^{-1}B^{-1}) = m-2$ special val. ± 2 $\Rightarrow m = 0, 4$

bonus: one can check that U_m is invariant under the outer action of $SL_2(\mathbb{Z})$ on F_2. So the condition that $\mathrm{tr}(ABA^{-1}B^{-1}) = k$ is "characteristic".

S_m is the closure of U_m in $\mathbb{P}^3$, with 3 rational lines at infinity. ($xyz = 0$ in $\mathbb{P}^2$) 

Random Shower Question: is the action of $SL_2(\mathbb{Z})$ on an adic neighborhood of  faithful?

Do they have $\mathbb{Q}$-points? They always do! in fact S_m is uni-rational

② Strong / Weak approximation

$\mathcal{U}/\mathbb{Z}$ a scheme.

$$\mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet = \prod_p \mathcal{U}(\mathbb{Z}_p) \times \pi_0(\mathcal{U}(\mathbb{R}))$$

picture a pro system $\varprojlim_N \mathcal{U}(\mathbb{Z}/N\mathbb{Z}) \times \pi_0(\mathcal{U}(\mathbb{R}))$

$\vdots \rightarrow \vdots$

sol set mod 12 $\qquad$ sol set mod 4

Strong approximation in adelic Language

$\mathcal{U}(\mathbb{Z})$ dense in $\mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet$

Practical Meaning:

For every $N > 0$, and any solution $x \in \mathcal{U}(\mathbb{Z}/N\mathbb{Z})$

If x comes from an adelic point (locally liftable, easy for smooth places) What happens at singular places? Can we guarantee liftability if N is large over these places?

Then there exists a global integral solution which reduces to x! This allows us to guess integral solutions?

Week approximation

the same practical meaning, except that we now allow denominators when finding a global solution which reduces to x.

E.g. Strong approximation is true for $\mathbb{A}^1$, false for $\mathbb{G}_m$ over any number ring.

(Reason: "Mordell-Weil" for $\mathbb{G}_m$)

BM-set can prevent $\mathcal{U}(\mathbb{Z})$ becoming dense in $\mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet$

$$\overline{\mathcal{U}(\mathbb{Z})} \subseteq \mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet^{Br} \subseteq \mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet$$

Something is going on here! This Br we use has to come from $\mathcal{U}/\mathbb{Q}$, otherwise it's a very boring set!

Is strong approximation useful for finding integral points on homogeneous varieties?

e.g. $x^2+y^2=n$

$$\mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet^{Br} = \mathcal{U}(\mathbb{A}_\mathbb{Q})_\bullet^{Br} \cap \mathcal{U}(\mathbb{A}_\mathbb{Z})_\bullet$$

③ Computing the Brauer-Manin group, in general.

General strat: Galois descent spectral sequence for $R\Gamma(\mathrm{Gal}, H^*(X^s, \mathbb{G}_m))$ (Hochschild-Serre spectral sequence) ($\mathrm{Br}_1 X := \ker(\mathrm{Br}X \to \mathrm{Br}\overline{X})$)

$$0 \longrightarrow H^1(k, k_s[X]^*) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^s)^\Gamma \longrightarrow H^2(k, k_s[X]^*) \longrightarrow \mathrm{Br}_1(X)$$
$$\longrightarrow H^1(k, \mathrm{Pic}(X^s)) \longrightarrow \mathrm{Ker}[H^3(k, k_s[X]^*) \to H^3_{\text{ét}}(X, \mathbb{G}_m)].$$

Every arrow has a meaning! Put this up on a wall!

Fact: For number fields $H^3_{et}(k, k_s^*) = 0$. I don't know a proof yet.

This kind of spectral seq. calculation is why Grothendieck's cohomological definition is superior! To achieve a true understanding of this sequence, one will need to understand these transfer maps.

E.g. for $X = \mathbb{A}^1, \mathbb{P}^1 / k$, $Br\, k \to Br\, X$ is iso!

$Br\, \mathbb{G}_m / Br\, k \simeq H^2_{et}(k, \mathbb{Z}) \simeq Hom(Gal(k), \mathbb{Q}/\mathbb{Z}) = \underbrace{Hom_{cont.}(\hat{\mathbb{Z}}^\times, \mathbb{Q}/\mathbb{Z})}_{\text{explicitly?}}$

This contains info. about every cyclic Galois extension. $k = \mathbb{Q}$

Question: does BM explain failure of SA for $\mathbb{G}_m$?

Fact: $Br\, \bar{X}$ can be computed more "easily" for sm. proj $\bar{X}$.

$$0 \to (\mathbb{Q}/\mathbb{Z})_\ell^{b_2 - \rho} \to Br\, \bar{X}_{\ell\text{-primary}} \to H^3_{et}(\bar{X}, \mathbb{Z}_\ell(1))_{tor} \to 0$$

$b_2 = \text{rank } H^2_{et}(\bar{X}, \boxed{\mathbb{Z}_\ell(1)})$ $\qquad (\mathbb{Q}/\mathbb{Z})_\ell^{b_2-\rho} := \left(\frac{H^2_{et}(\bar{X}, \mathbb{Z}_\ell(1))}{NS(\bar{X}) \otimes \mathbb{Z}_\ell} \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell / \mathbb{Z}_\ell$

$\rho = \text{rank } NS(\bar{X}) \otimes \mathbb{Z}_\ell$

Tate conjecture: the subspace (inverting ℓ) (Known for K3 now!)

$NS(\bar{X}) \otimes \mathbb{Q}_\ell \subset H^2_{et}(\bar{X}, \mathbb{Q}_\ell(1))$

can be read-off from the Galois action on $H^2_{et}(\bar{X}, \mathbb{Q}_\ell(1))$

$\Rightarrow$ $Br\, \bar{X}_{\ell\text{-primary}}$ is determined by ℓ-adic étale cohomology of $\bar{X}$ (with Galois representation considered)

More fun fact: Due to birational invariance of $Br\, \bar{X}$ the torsion part of $H^3_{sing}(\bar{X}(\mathbb{C}), \mathbb{Z})$ is a birational invariant!

Some final observation:

A lot of information in $Br_1 X$ in the 6-term ext. seq. comes from $Pic\, X$ (and $Pic\, \bar{X}$)

Thus these info is captured a lot by the "Albanese torsor" $u_X : X \to Alb^1 X$.

Therefore to understand Brauer-Manin obstruction better, one should first understand it for abelian torsors.

④ Computing the Brauer-Manin group for S_m

Spectral seq says for S_m, we have $\operatorname{Br} S_m / \operatorname{Br} k \xrightarrow{\sim} H^1_{ét}(k, \operatorname{Pic} S_m^s)$

It's possible to compute, and we can already obtain a tight upper bound on $H^1_{ét}(K, \operatorname{Pic} S_m^s)$.

The idea is beautiful, represent $\operatorname{Pic} S_m^s$ as a quotient of the Galois modules generated by the 27-lines on a cubic surface. The Galois monodromy must factor through the geometric monodromy over the moduli space of all cubic surfaces, which is a permutation representation of $W(E_6)$!!

4 dim

Now one can brute-force it and obtain upper bound on $\operatorname{Br} S_m$.

(A little more details here, we just need to focus on possible Galois monodromies which give us 3 coplanar lines defined over $\mathbb{Q}$, this eliminates the search by a lot). This computation is used to upper bound $\operatorname{Br} \mathcal{U}_m$ in [LM].

④$^+$ In fact we can do better!

Let's pick one line at infinity on S_m, and consider the $\mathbb{P}^1$-pencil of planes going through it (basically, projecting from a generic line to this line). This turns S_m into a conic bundle:

$$S_m: \quad w(x^2+y^2+z^2) - xyz = mw^3, \quad (x:y:z:w) \in \mathbb{P}^3$$

proj onto $(x:w) = (t:s) \in \mathbb{P}^1$

$$S_m/k(\mathbb{P}^1): \quad \left(\frac{t}{s}\right)^2 + Y^2 + Z^2 - \frac{t}{s}YZ = m \iff \left(Y - \frac{t}{2s}Z\right)^2 + \left(1 - \frac{t^2}{4s^2}\right)Z^2 - \left(m - \frac{t^2}{s^2}\right) = 0.$$

the class for S_m in $\operatorname{Br} k(\mathbb{P}^1)$ is $\quad \mathcal{E} = \left(\frac{t^2}{4s^2} - 1,\ m - \frac{t^2}{s^2}\right)$

Ramification locus $\longrightarrow$ $s \cdot (4s^2 - t^2) \cdot (t^2 - ms^2) = \Delta$

Locations: $\frac{s}{t} = 0, -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{\sqrt{m}}$ defined over $k(\sqrt{m})$

You might recognize these numbers from my donut talk.

Let $\mathcal{U} = \mathbb{P}^1 \setminus \Delta$, $k(\mathcal{U})$ be the generic point of $\mathcal{U}$. The conic bundle $S_{m/k(\mathcal{U})} \to \operatorname{Spec} k(\mathcal{U})$ induces an exact sequence.

$$0 \to \underset{\substack{\text{IS} \\ \text{the class of } S_{m/k(\mathcal{U})}}}{\langle \mathcal{E} \rangle} \to \operatorname{Br} k(\mathcal{U}) \to \operatorname{Br} S_{m/k(\mathcal{U})} \to 0$$

Since $\operatorname{Br} S_m \subset \operatorname{Br} S_{m/k(\mathcal{U})}$, this means every class in $\operatorname{Br} S_m$ is represented by the pull back of a class in $\operatorname{Br} k(\mathcal{U})$. In order to understand which class in $\operatorname{Br} k(\mathcal{U})$, we need to understand the local behavior of a class in DVRs of both $k(\mathcal{U})$ and $k(S_m)$.

Local structure: Let R be a DVR, K be the fraction field, k_v be the residue field. $\operatorname{Spec} K \xhookrightarrow{j} \operatorname{Spec} R \xhookleftarrow{i} \operatorname{Spec} k_v$. Then $R^{\geq 1} j_* \mathbb{G}_m = 0$, and we have an exact sequence.

$$1 \to \mathbb{G}_m \to R^0 j_* \mathbb{G}_m \xrightarrow{v} i_* \mathbb{Z} \to 0$$

Inducing

$$0 \to H^2(R, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m) \to \underset{\substack{\text{IS} \\ H^1(k_v, \mathbb{Q}/\mathbb{Z})}}{H^2(k_v, \mathbb{Z})} \to \cdots$$

with $\partial : H^2(K, \mathbb{G}_m) \to H^1(k_v, \mathbb{Q}/\mathbb{Z})$.

Definition: Let $c \in H^2(K, \mathbb{G}_m)$, the image $\partial(c)$ in $H^1(k_v, \mathbb{Q}/\mathbb{Z})$ is call the residue of c.

Practically: $\partial(c)$ is a cyclic extension of k_v associated to c.
if c is given by a Hilbert symbol $(t^a u, t^b v)_K$ where t is the uniformizer.

$$\begin{aligned} (t^a u, t^b v) &= (t,t)^{ab} (t,v)^a (u,t)^b (u,v) \\ &= (t, (-1)^{ab}) (t, v^a u^b) (u,v) \\ &= (t, (-1)^{ab} u^b v^a) \underbrace{(u,v)}_{\text{integral.}} \end{aligned}$$

if $\operatorname{char}(k_v) \neq 2$

Then $\partial_v((t^a u, t^b v))$ is $(-1)^{ab} u^b v^a \in k_v^\times / (k_v^\times)^2$.

i.e. the quadratic extension is given by $\sqrt{(-1)^{ab} u^b v^a}$.

Definition: Let X be a smooth scheme $/K$, with function field $K(X)$. The unramified Brauer group:

$$\mathrm{Br}_{un}(K(X)) := \{ c \in \mathrm{Br}(K(X)) \mid \partial_v(c) = 0 \ \forall v \text{ d.v. of } K(X) \}$$

Question: for any $c \in \mathrm{Br}(K(X))$, are there finitely many v s.t. $\partial_v(c) \neq 0$. ?? This is serious.

if c is ramified over a normal crossing X

what happens in the blow up?

Rmk: in the curve case we have such finiteness.

Definition: Let $c \in \mathrm{Br}\, K(X)$, a class c' is subordinate to c ($c' \leq c$) if for all v. d.v. of $K(X)$

$$\partial_v(c') \in \langle \partial_v(c) \rangle \subset H^1(k_v, \mathbb{Q}/\mathbb{Z})$$
$$(c' \leq_v c)$$

Meaning the cyclic field $\partial_v(c')$ is a subfield $\partial_v(c)$

The group formed by such c' is denoted $\mathrm{Sub}(K(X), c)$

Rmk: in the curve case, there are <u>only finitely many $c' \leq c$.</u>
due to finiteness of number of d.v.r. with non-zero residue for c.

Final computation: In our situation

S_m $\qquad$ X $\quad$ X $\quad$ X $\quad$ X $\qquad$ not every node is split

$\downarrow$

$(s:t) = \mathbb{P}^1 \ni \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2} \quad \pm\frac{1}{\sqrt{m}}$ $\qquad$ P_1 splits

$\qquad\qquad\qquad\quad P_1 \quad P_2 \quad P_3 \quad P_4$ $\qquad$ P_2, P_3, P_4 splits when added $\sqrt{m-4}$

$\mathcal{U} = \mathbb{P}^1 \setminus \{P_1, P_2, P_3, P_4\}$, $\quad S_m/k(\mathcal{U}) \rightsquigarrow \varepsilon = \left(\frac{t^2}{4s^2} - 1,\ m - \frac{t^2}{s^2}\right) \in \mathrm{Br}\,k(\mathcal{U})$

In the end we will produce every class in $\mathrm{Br}\, S_m$ using $\mathrm{Sub}(k(\mathbb{P}^1), \varepsilon)$

I. computing the local residues

$\partial_{P_1}(\mathcal{E})$: $\left(\frac{t^2}{4s^2}-1,\ m-\frac{t^2}{s^2}\right) = \left(\frac{1}{4}-\frac{s^2}{t^2},\ \frac{s^2}{t^2}m-1\right)$

is unramified! But $S_m \to \mathbb{P}^1$ is singular at P_1?

This is because this singularity can be resolved via blowing down a line in the fiber above P_1.

Remember that all lines on a cubic surface are (-1)-curves.

$\Rightarrow\ \partial_{P_1}(\mathcal{E}) = 0.$

$\partial_{P_2}(\mathcal{E})$: $\left(\underbrace{\left(\frac{1}{2}+\frac{s}{t}\right)}_{\text{uniformizer}}\left(\frac{1}{2}-\frac{s}{t}\right),\ \frac{s^2}{t^2}m-1\right) = \left(\left(\frac{1}{2}+\frac{s}{t}\right),\ \frac{s^2}{t^2}m-1\right)\cdot\underbrace{\left(\frac{1}{2}-\frac{s}{t},\ \frac{s^2}{t^2}m-1\right)}_{\text{unramified}}$

$\Rightarrow$ Quadratic extension generated by $\sqrt{m-4}$ over $\mathbb{Q}$

$\partial_{P_3}(\mathcal{E})$: similar as above, quadratic extension gen. by $\sqrt{m-4}$ over $\mathbb{Q}$

$\partial_{P_4}(\mathcal{E})$: quadratic extension gen. by $\sqrt{m-4}$ over $\mathbb{Q}(\sqrt{m})$

II. compute $\mathrm{Sub}(k(\mathbb{P}^1), \mathcal{E})$

$$0 \to \underset{\underset{\mathrm{Br}\,\mathbb{Q}}{\wr\wr}}{\mathrm{Br}\,\mathbb{P}^1} \to \underset{\underset{\mathrm{Br}\,\mathbb{Q}(\mathbb{P}^1)}{\cap}}{\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1),\mathcal{E})} \xrightarrow{\partial_{P_2},\partial_{P_3},\partial_{P_4}} (\mathbb{Z}/2)^3$$

2-torsion!

Note that all classes in $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1),\mathcal{E})$ are unramified over $W = \mathbb{P}^1 \setminus \{P_2, P_3, P_4\}$ but this fact alone won't show finiteness of $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1),\mathcal{E})$.

Note that every class $c \in \mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1),\mathcal{E})$ becomes unramified after adjoining $\sqrt{m-4}$ to the field of constants $\mathbb{Q}$.

finite etale cover

$$W(\sqrt{m-4}) \xrightarrow[\pi]{} W \qquad R^{\geq 1}\pi_*\mathbb{G}_m = 0$$

etale sheaves $1 \to \mathbb{G}_m \to \pi_*\mathbb{G}_m \xrightarrow{\sigma-1} N \to 1$ over W.

$\Rightarrow\ \ker\left(H^2(\mathbb{G}_m) \to H^2(\pi_*\mathbb{G}_m)\right) \cong H^1(W, N)$

all Br classes which split on $W(\sqrt{m-4})$

$N = \{x^2-(m-4)y^2=1\}$ is the norm torus in $\mathbb{Q}(\sqrt{m-4})$.

We will need to understand $H^1(W,N)_2 / H^1(\mathbb{Q},N)_2$

Lemma. The set $N(W) := \{ X^2 - (m-4) Y^2 = 1,\ X, Y \in \mathbb{Q}[W] \}$ only contains constants. i.e. $N(W) = N(\mathbb{Q})$.

Pf. Over $\bar{\mathbb{Q}}$ we have the exact sequence

$$0 \to N(\bar{\mathbb{Q}}) \to N(\bar{W}) \to L(1) \to 0$$

$L \subset \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Res}^{\sqrt{m}} \mathbb{Z}$ records zeroes and poles at $\pm 1/2$, $\pm 1/\sqrt{m}$

$\mathrm{Res}^{\sqrt{m}} \mathbb{Z}$ is $\mathbb{Z}^2$ where $\mathrm{Gal}(\mathbb{Q}\sqrt{m}/\mathbb{Q})$ acts as permutation.

$L(1)$ means Galois action from $\sqrt{m-4}$ acts by ± 1 on L.

$$L(1)^\Gamma \subset \left(\mathbb{Z}(1) \oplus \mathbb{Z}(1) \oplus \mathrm{Res}^{\sqrt{m}} \mathbb{Z}(1) \right)^\Gamma = 0.$$

$$\mathbb{Z}(1)^\Gamma = 0. \qquad \left(\mathrm{Res}^{\sqrt{m}} \mathbb{Z}(1) \right)^\Gamma = 0.$$

$$\Rightarrow \quad N(W) = N(\mathbb{Q})$$

Corollary: We have a natural identification

$$H^1(W, \{\pm 1\}) / H^1(\mathbb{Q}, \{\pm 1\}) \simeq H^1(W, N)_2 / H^1(\mathbb{Q}, N)_2$$

Thus the non-constant 2-torsion Brauer classes on W which split on $W(\sqrt{m-4})$ corresponds exactly to geometrically connected double covers of W.

Pf:

On W we have exact sequence

$$0 \to \{\pm 1\} \to N \xrightarrow{2} N \to 1$$

Take $H^*(W, -)$ and apply previous lemma.

$$\begin{array}{ccccccccc} 1 \to & N(W)/N(W)^2 & \to & H^1(W, \pm 1) & \to & H^1(W, N)_2 & \to 0 \\ & \uparrow \wr & & \uparrow & & \uparrow & \\ 1 \to & N(\mathbb{Q})/N(\mathbb{Q})^2 & \to & H^1(\mathbb{Q}, \pm 1) & \to & H^1(\mathbb{Q}, N)_2 & \to 0 \end{array}$$

Consequently, $H^1(W, N)_2$ encodes geometrically connected double covers of W. Now apply Kummer theory

$$1 \to \{\pm 1\} \to \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m \to 1$$

$$\Rightarrow \quad 1 \to \mathbb{Q}^\times / \mathbb{Q}^{\times 2} \to \mathbb{Q}[W]^\times / \mathbb{Q}[W]^{\times 2} \to H^1(W, N)_2 / H^1(\mathbb{Q}, N)_2 \to 1$$

$\circ \quad \circ \quad \circ^2$

$-\frac{1}{2} \quad \frac{1}{2} \quad \pm\frac{1}{\sqrt{m}}$

$-1, -1, 1$

$-1, 1$

Generators of $\mathbb{Q}[W]^\times$

① $(4s^2-t^2)^{-1} \cdot (ms^2-t^2)$

② $(2s-t)^{-1} \cdot (2s+t)$

These will generate our desired Brauer classes.

Next, we convert these generators back to classes in $\mathrm{Br}\,W$.

We want to turn an N-torsor on W to a conic over W. Observe that N is already an affine conic, it can be guessed that any twist of N by a class in $H^1(W, \{\pm1\})$ should also produce a conic, we just lose the identity section. (Note that everything can be checked over the generic point of W)

Lemma / Observation Let K be a char $\neq 2$ field, $a \in K^\times$, $N = \{x^2 - ay^2 = 1\}$ an algebraic torus. Let $L = K(\sqrt{d})$ for some $d \in K^\times$.
Then the L-twist of N is the affine conic

$$N_L := \{x^2 - ay^2 = d\} \text{ over } K. \sim (a, d)_K$$

Note that $N_L = (\mathrm{Spec}\, L \times_K N)/(\mathbb{Z}/2\mathbb{Z})$ by definition, where the involution acts on $\mathrm{Spec}\, L$ as $(\sqrt{d} \mapsto -\sqrt{d})$ and it acts on N as translation by $(-1, 0)$.

Pf:

$$N_L = \mathrm{Spec}\left(K[x, y, \sqrt{d}]/(x^2 - ay^2)\right)^{\sigma\text{-inv.}}$$

$$\sigma : x \mapsto -x,\ y \mapsto -y,\ \sqrt{d} \mapsto -\sqrt{d}$$

Define $u = x\sqrt{d}$, $v = y\sqrt{d}$, then u and v are σ-inv.

Then $K[x, y, \sqrt{d}]/(x^2 - ay^2 - 1)$ is degree 2 over

$$K[u, v]/(u^2 - av^2 - d).$$

Rmk: One then needs to check this way of producing a conic is compatible with the transfer map $H^1(K, N)_2 \to H^2(K, \mathbb{G}_m)_2$. Thankfully we don't need to worry about signs here.

The non-constant classes in $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1), \mathcal{E})$ are

① $\left((4\frac{s^2}{t^2}-1)\cdot(m\frac{s^2}{t^2}-1),\ m-4\right) = \mathcal{E}$

② $\left(4\frac{s^2}{t^2}-1,\ m-4\right)$

Note that the class ① has the same residues over P_2, P_3, P_4 as $\mathcal{E}$, so they are equal.

III. Showing $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1), \mathcal{E})$ generate everything in $\mathrm{Br}\, S_m$

$$\begin{array}{ccccccc} S_m & & X & X & X & X \\ \downarrow & & & & & \\ (s:t) = \mathbb{P}^1 & \ni & 0 & -\frac{1}{2} & \frac{1}{2} & \pm\frac{1}{\sqrt{m}} \\ & & P_1 & P_2 & P_3 & P_4 \end{array}$$

not every node is split

P_1 splits, and $\mathcal{E}$ is unramified at P_1

P_2, P_3, P_4 splits when added $\sqrt{m-4}$

First, the pull-back of a class $c \in \mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1), \mathcal{E})$ to S_m is unramified. We can check this by computing local residues on S_m.

① Let v be a DVR corresponding to a horizontal curve (dominating $\mathbb{P}^1$ below) then the pull-back of c is clearly unramified (as it is generically unramified on this horizontal curve)

② Let v be a DVR corresponding to a vertical curve. It suffices to consider curves getting mapped down to P_2, P_3 or P_4.

Let's say v is above P_2, let R be the DVR of P_2, let R' be the DVR of v. The local residues are compatible with DVR extensions $R \subset R'$ in the following way, $e(R'/R)$ is the ramification between valuations.

$$\begin{array}{ccc} H^2(K', \mathbb{G}_m) & \xrightarrow{\partial_{R'}} & H^1(R'_v, \mathbb{Q}/\mathbb{Z}) \\ \uparrow \mathrm{Res} & & \uparrow e(R'/R)\cdot \mathrm{Res} \\ H^2(K, \mathbb{G}_m) & \xrightarrow{\partial_R} & H^1(R_v, \mathbb{Q}/\mathbb{Z}) \end{array}$$

Ramification unwinds cyclic extensions.

Now since $c \leq_{P_2} \mathcal{E}$, and the pull-back of $\mathcal{E}$ to S_m is trivial, we see that c is unramified on S_m.

Second, let's show every class on $\mathrm{Br}\, S_m$ comes from $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1), \varepsilon)$.

We already know every class on $\mathrm{Br}\, S_m$ comes from $\mathrm{Br}\, \mathbb{Q}(\mathbb{P}^1)$, it now suffices to show that if a class $c \in \mathrm{Br}\, \mathbb{Q}(\mathbb{P}^1)$ does not lie in $\mathrm{Sub}(\mathbb{Q}(\mathbb{P}^1), \varepsilon)$, then it acquires a non-zero residue over certain curve on S_m.

Without loss of generality, we may assume c is ramified at one of the P_i, and that $\partial_{P_i}(c) \notin \langle \partial_{P_i}(\varepsilon) \rangle$. This means $\partial_{P_i}(c) \neq 0$ even if we adjoin $\sqrt{m-4}$ to $\mathbb{Q}$. But then the node in the fiber of $S_m \to \mathbb{P}^1$ becomes split over the residue field of P_i. Pick the DVR v on S_m corresponding to one of the line in the fibre. The local ring extension above this P_i looks like

$(K = \mathbb{Q}(P_i)(\sqrt{m-4}))$

$$\begin{array}{lccc} \text{DVR} & K[x]_{(x)} & \longrightarrow & K(y)[x]_{(x)} \\ & \downarrow & & \downarrow \\ \text{residue field} & K & \longrightarrow & K(y) \end{array}$$

Now since the ramification index is one, the local residue of c at v is the cyclic extension coming from pull-back a cyclic extension of K. The extension will remain non-split since $\mathbb{P}^1$ is geometrically connected.

Conclusion: There is only one non-trivial element in $\mathrm{Br}\, S_m$, it is given by

$$\left(4\frac{s^2}{t^2} - 1,\ m-4\right)$$