Locally constant torsors ≈ "covering spaces"

↳ "universal cases, e.g. universal covering space is a torsor by the fun group".

Smooth / connected torsors = "Principal bundles / principal homogeneous space"

Reduction of structure group (a common theme)

$P \times_H G$

X connected finite type scheme$/\mathbb{Z}$

Review of sheaves: Any morphism $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ can be thought of as a sheaf

$$U \subset X \mapsto P(U) = \left\{ \text{sections } \begin{tikzcd} & P \arrow[d] \\ U \arrow[ur] \arrow[r, phantom, "\subset"] & X \end{tikzcd} \right\}$$

such kind of sheaf is said to be "representable"

E.g. $G = \mathbb{Z}/n\mathbb{Z}$ the constant group, $P = X \times G \xrightarrow{p_1} X$

represent locally constant functions on X valued in $\mathbb{Z}/n\mathbb{Z}$.

$G = GL_n$ an algebraic group, $P = X \times G \xrightarrow{p_1} X$

algebraic functions, they can vary.

represent matrix valued functions $\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \ddots & & \vdots \\ f_{n1} & \cdots & & f_{nn} \end{pmatrix}$ which is locally invertible (over all points)

Def: A torsor for a group scheme $\begin{matrix} G \\ \downarrow \\ X \end{matrix}$ is a (representable) sheaf $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$

such that P is locally isomorphic to G, in a "not necessarily compatible" way.

i.e. there exists a cover $\varphi : \coprod_i U_i \to X$ and an isomorphism $\varphi^* P \xrightarrow{\sim} \varphi^* G$.

Furthermore, we want these local isomorphisms only differ by a left translation on G.

$$\begin{tikzcd} P_{ij} \arrow[r, "\varphi_i"] \arrow[dr, "\varphi_j"'] & G_{ij} \arrow[d, "L_{g_{ij}}"] \\ & G_{ij} \end{tikzcd}$$

← it's important that these local differences is only a left translation!

Rmk: If these local isomorphisms were compatible, then P is globally isomorphic to G!

Observation ①: There exists a global right $\mathcal{G}$-action on P s.t. the following diagram
(Homogenity) is fibered (check locally)

$$\begin{array}{ccc} P \times \mathcal{G} & \xrightarrow{\alpha} & P \\ p_1 \downarrow & & \downarrow p \\ P & \xrightarrow[p]{} & X \end{array}$$

(the action map)

To glue the right $\mathcal{G}$-action, take $g \in G$ and do $\varphi_i^{-1} R_g \varphi_i$. These local translations glue because $\varphi_i^{-1} R_g \varphi_i = \varphi_j^{-1} R_g \varphi_j$, as $\varphi_j \varphi_i^{-1} = L_{ij}$ commutes with R_g

Observation ②:
(1-cocycle) These local incompatibilities define a 1-cocycle valued in the sheaf of groups $\mathcal{G}$.

Furthermore, two different local trivializations differ by a "1-coboundary"

$P_i \xrightarrow{\varphi_i} \mathcal{G}_i$ v.s. $P_i \xrightarrow{\varphi_i'} \mathcal{G}_i$ differ by a translation

$h_i = \varphi_i' \varphi_i^{-1}(1)$

$$\begin{array}{ccc} P_i & \xrightarrow{\varphi_i} & \mathcal{G}_i \\ & {\scriptstyle \varphi_i'} \searrow & \downarrow L_{h_i} \\ & & \mathcal{G}_i \end{array}$$

Conversely, given a 1-cocycle in $\mathcal{G}$ for some cover of X, we can produce a torsor P via sheaf gluing. ⚠

$$\mathcal{G}_i|_{U_{ij}} \underset{L_{ij}}{\xrightarrow{\sim}} \mathcal{G}_j|_{U_{ij}}$$

$$\begin{array}{ccc} \mathcal{G}_i|_{U_{ijk}} & \xrightarrow{L_{ik}} & \mathcal{G}_k|_{U_{ijk}} \\ {\scriptstyle L_{ij}} \searrow & & \nearrow {\scriptstyle L_{jk}} \\ & \mathcal{G}_j|_{U_{ijk}} & \end{array}$$

Can check that this construction produces isomorphic sheaves if the 1-cocycle g_{ij} is replaced by $h_i g_{ij} h_j^{-1}$.

The take-away: A torsor P for a group scheme $\mathcal{G}$ is a (scheme) a sheaf of right $\mathcal{G}$-sets → equipped with a "transitive" homogeneous right $\mathcal{G}$-action. Isomorphism classes of $\mathcal{G}$ torsors is given by sheaf cohomology $\check{H}^1(X, \mathcal{G})$.

More abstract non-sense When $\mathcal{G}$ is abelian, $\check{H}^1(X, \mathcal{G})$ acquires a group structure when $[P_1] - [P_2]$ is represented by the isomorphism sheaf $\mathcal{I}so_{\mathcal{G}}(P_2, P_1)$.

Reduction of structure group For a homomorphism $\mathcal{G} \to \mathcal{H}$, the induced map $\check{H}^1(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{H})$ is given by

$$P \mapsto P \times^{\mathcal{G}} \mathcal{H} = (P \times \mathcal{H} / \mathcal{G}) \text{ as sheaves}$$

$$p \cdot h \mapsto pg^{-1}, gh$$

More generally:

Twists Let $\begin{matrix} \mathcal{F} \\ \downarrow \\ X \end{matrix}$ (a sheaf) carry a left $\mathcal{G}$-action, a twisted form of $\mathcal{F}$ by a class $[P] \in \check{H}^1(X, \mathcal{G})$ is given by $P \times^{\mathcal{G}} \mathcal{F}$.

A twist is locally isomorphic to $\mathcal{F}$, but we allow ambiguity in $\mathcal{G}$ when gluing back.

usually resulting from some extra structural symmetry.

Examples of twists and torsors: coherent version

① Twists of the "schematic" vector space $\mathbb{V}_n$ (has addition and scalar multiplication)

Vector bundle

gluing data lies in the automorphism group ~~scheme~~ sheaf $GL_n = GL(\mathbb{V}_n)$ (preserves)

Let $\begin{matrix} \mathcal{E} \\ \downarrow \\ X \end{matrix}$ be a twist, let $\mathcal{F}(\mathcal{E})$ be the "frame bundle".

$$\mathcal{F}(\mathcal{E})(U) := \{(e_1, \dots, e_n) : \mathcal{O}(U)\text{-basis of } \mathcal{E}(U)\}$$

this is a sheaf, a GL_n-torsor in fact.

The right $GL_n(U)$ action is given by $(e_1, \dots, e_n)\begin{pmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & \ddots & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{pmatrix}$

Conversely for a GL_n-torsor P, there is a twisted form $P \times^{GL_n} \mathbb{V}_n$, which still has addition and scalar multiplication, and is locally isomorphic to $\mathbb{V}_n$.

Slogan: (coherent) vector bundles are twisted forms of $\mathbb{V}_n$, and is classified by GL_n-torsors.

Bonus $\check{H}^1_{Zar}(X, GL_n) \to \check{H}^1_{ét}(X, GL_n) \to \check{H}^1_{fppf}(X, GL_n)$ are isomorphisms!

$$\check{H}^1_{ét}(X, GL_n) \cong \check{H}^1_{sm}(X, GL_n)$$

② Azumaya algebra (cheat version)

An Azumaya algebra is a coherent sheaf of $\mathcal{O}_X$-algebra which is étale locally isomorphic to the matrix algebra $M_n(\mathcal{O}_X)$. cheating!

Since $\mathrm{Aut}(M_n(\mathcal{O}_X))$ is the group scheme PGL_n, the gluing data lies in PGL_n.

②' Severi-Brauer variety

Associated to an Azumaya algebra is another sheaf which happens to be a twisted form of $\mathbb{P}^{n-1}$.

The observation: if an Azumaya algebra can act non-trivially on V_n, then this Azumaya alg. is the un-twisted one. Instead of thinking about abstract representations, we can think about left-ideals corresponding to such a representation.

Def: The Severi-Brauer variety $S(A)$ is a sheaf of left-ideals of A:

$$S(A)(U) = \{ I \subset A|_U \mid \text{s.t. } A/I \ \underbrace{\text{locally free of rank } n}_{\text{instead of } V_n} \}$$

One can check that this is indeed a sheaf.

Thm. A Severi-Brauer variety is a twisted form of $\mathbb{P}^{n-1}$, and all twists of $\mathbb{P}^{n-1}$ arises in this way. In particular, if $\mathcal{E} \downarrow X$ is a vector bundle of rank n, then $A = \mathrm{End}(\mathcal{E})$ is an Azumaya algebra and

$$S(A) = \mathbb{P}(\mathcal{E}^\vee) \ (\text{or } \mathbb{P}(\mathcal{E})\ ?)$$

②'' Follow-up of Omega Tong and there is a surprise!

Analogous to frame bundles in the vector bundle example, here one may construct the torsor from a twist abstractly. Let $S = S(A)$ be a Severi-Brauer variety. The isomorphism sheaf $\underline{\mathrm{Iso}}(\mathbb{P}^{n-1}, S)$ is the PGL_n-torsor associated to S in a purely abstract way.

One might hope to produce a concrete torsor just like the frame bundle, and we have the following observation:

Observation the action of PGL_n on $\mathbb{P}^{n-1}$ is $(n+1)$-transitive and free. More precisely, the open subscheme of "general $(n+1)$-points"

$$G_{n+1}(\mathbb{P}^{n-1}) = (\mathbb{P}^{n-1})^{n+1} - \mathrm{Disc} \quad . \quad \mathrm{Disc} = \prod_{i=1}^{n+1} \det(M_i) \; , \; M_i \text{ are } n\times n \text{ minors of } \begin{bmatrix} | & | & | & & | \\ P_1 & P_2 & P_3 & \cdots & P_{n+1} \\ | & | & | & & | \end{bmatrix}$$

is a left PGL_n-torsor.

However I'm having trouble showing that this construction produces a PGL_n-torsor. The obstacle is, the obvious construction of $G_{n+1}(S)$ comes from twisting on the left, where $P \in \check{H}^1(X, PGL_n)$ represents the gluing for S.

$$G_{n+1}(S) = P \times^{PGL_n} G_{n+1}(\mathbb{P}^{n-1})$$

But we can't glue the local left torsor structure back.

Maybe more in principal, the scheme $G_{n+1}(\mathbb{P}^{n-1})$ does not represent the functor $\underline{\mathrm{Iso}}(\mathbb{P}^{n-1}, \mathbb{P}^{n-1})$. I think the pointed scheme $G_{n+1}(\mathbb{P}^{n-1}) \ni \left(\begin{smallmatrix} 1 & 0 & & & 1 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & & & 1 & 1 \end{smallmatrix} \right)$ does.

On the other hand, the functor $\underline{\mathrm{Iso}}(S, \mathbb{P}^{n-1})$ has a more explicit description using the functorial interpretation of $\mathbb{P}^{n-1}$.

But to be fair, I'm not entirely sure if Omega's approach really does not work. The construction must say something interesting.

More examples of twists and torsors: locally constant version

X geometrically connected over $\mathbb{Q}$ (or $\mathbb{Q}_p$), with <u>a rational base point $b \in X$</u> chosen.

this is a non-obvious restriction in our set-up.

Geometric universal cover

This is more or less the universal example of locally constant torsors. First we show that there exists a family of geometrically connected finite etale covers of X which is co-final over $\overline{\mathbb{Q}}$.

Construction: Let $(Y, \hat{b}) \to (\overline{X}, b)$ be a finite etale cover over $\overline{\mathbb{Q}}$. Let $Z = \prod Y^\sigma$ fibered over $\overline{X}$, a product of Galois conjugates of Y. Then the connected component of the point $(\hat{b}^\sigma)_\sigma$ will descend back to $\mathbb{Q}$.

I think I've seen this somewhere, probably Serre.

Rmk: Dually, what we are doing here can be explained using étale fundamental groups as follows.

$$1 \to \pi_1(\bar{X}, b) \to \pi_1(X, b) \underset{\curvearrowleft s}{\to} \mathrm{Gal}\,\mathbb{Q} \to 1$$

s coming from our base point b.

Let $H \subset \pi_1(\bar{X}, b)$ be a finite index subgroup, then $\check{H} := \bigcap H^{\sigma}$ is finite index, and is invariant under $s(\mathrm{Gal}\,\mathbb{Q})$ actions. Now the subgroup $s(\mathrm{Gal}\,\mathbb{Q})\check{H}$ in $\pi_1(X, b)$ will have the same index as $\check{H}$ in $\pi_1(\bar{X}, b)$. This is the geometrically connected cover we want.

Definition: Let $(Y_i, b_i) \to (X, b)$ be a cofinal family of geometrically connected finite étale cover, where b_i are geometric points on Y which map to b. Then the pro-system $(\hat{X}, \hat{b}) = \varprojlim_i (Y_i, b_i) \to (X, b)$ is called the geometric universal cover of (X, b).

Remark: Think of the geometric fundamental group $\pi_1(\bar{X}, b)$ as a (pro) locally constant group over $\mathbb{Q}$. Then the universal cover $(\hat{X}, \hat{b}) \to (X, b)$ is a (pro)-torsor for $\pi_1(\bar{X}, b)$.

Examples:

① $\varprojlim_n (\mathbb{G}_m, 1) \to (\mathbb{G}_m, 1)$ under $n : (\mathbb{G}_m, 1) \to (\mathbb{G}_m, 1)$ is a $\hat{\mathbb{Z}}(1)$-torsor.

each $\mathbb{Q}$-point x on $\mathbb{G}_m$ is thus related to a $\hat{\mathbb{Z}}(1)$-torsor $(x^{1/n})_n$.

The field $\bigcup_n \mathbb{Q}(x^{1/n})$ is Galois only when $x = \pm 1$.

Over $\mathbb{Q}_p$, we can analyze the path torsor more carefully.

Let's denote the restriction of the universal cover to a rational point by $\int_b^x$

$$\int_b^x : \mathbb{G}_m(\mathbb{Q}_p) \to \varprojlim_n H^1(\mathbb{Q}_p, \mu_n)$$

$$\wr\wr$$

$$\mathbb{Q}_p^\times \to \varprojlim_n \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^n \qquad \left((\mathbb{Q}_p^\times)^m \subset (\mathbb{Q}_p^\times)^n \text{ for } n \mid m \right)$$

- if n varies in a prime-to-p direction, this is locally constant.

  e.g. Locally we are looking at $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{n'} \simeq \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^{n'}$ and everything is determined by the residue disk.

Prime-to-p part of the Path torsor has good reduction mod p.

- if n varies in the p-adic direction, this will get interesting, as

  $$(1+p\mathbb{Z}_p)/(1+p\mathbb{Z}_p)^{p^k} \simeq \mathbb{Z}/p^k$$

  the p-adic direction varies faithfully.

  One way to see this really clearly is through the logarithm map

$$\begin{tikzcd}
1+p\mathbb{Z}_p \arrow[r] \arrow[dr, "\log"'] & \varprojlim H^1(\mathbb{Q}_p, \mu_{p^n}) \\
& \mathbb{Z}_p = \varprojlim (1+p\mathbb{Z}_p)/(1+p\mathbb{Z}_p)^{p^n} \arrow[u, leftrightarrow]
\end{tikzcd}$$

Thus over the p-adics, the p-adic étale covers will vary interestingly, while the prime-to-p part is locally constant in p-adic topology.

② char p elliptic curve $E \xrightarrow{\phi^n-1} E$ is cofinal, where the transition is given by norm maps $N: E(\mathbb{F}_{q^n}) \to E(\mathbb{F}_{q^m})$ for $m \mid n$.

Note that these norm maps are surjective.

$$\begin{tikzcd}
E(\mathbb{F}_{q^n}) \arrow[r] \arrow[d, "N"] & E \arrow[r, "\phi^n-1"] \arrow[d, "\Sigma\phi^{mi}"] & E \arrow[d, equal] \\
E(\mathbb{F}_{q^m}) \arrow[r] & E \arrow[r, "\phi^m-1"] & E
\end{tikzcd}$$